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Statistical properties of water waves. Part 1. Steady-state distribution of wind-driven gravity–capillary waves

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The Miles–Phillips model of the linear coupling between waves on the ocean surface and a fluctuating wind field is generalized to include the average effect of the nonlinear water-wave interactions in the dynamic equations for gravity–capillary waves. A statistical-linearization procedure is applied to the general problem and yields the *optimum* linear description of the nonlinear terms by linear terms. The linearized dynamic equations are stochastic with solutions that have stable moments, i.e. the average nonlinear interactions quench the linear instability generated by the coupling to the mean wind field. In particular, an asymptotic steady-state power-spectral density for the water-wave field is calculated exactly in the context of the model for various wind speeds.

1. Introduction

The excitation of wave motion on the surface of a fluid by turbulent air flow over the surface is a long-standing problem in hydrodynamics. The physical mechanisms coupling the air and water must be understood in order to describe the evolution of the wind-generated spectrum of water waves on the ocean surface. Phillips (1957) proposed a stochastic model of the excitation mechanism in which the pressure field at the fluid surface is assumed to fluctuate independently of the surface response. These incoherent fluctuations drive the surface at length and time scales already extant in the pressure field spectrum. Miles (1957) proposed a deterministic mechanism involving the modulation of the air flow by the vertical movement of the surface, resulting in the pressure field doing work on the surface in-phase with the surface response. These complementary mechanisms were later synthesized by Miles (1960) into the Miles–Phillips model of inviscid resonant shear-flow instability. West & Seshadri (1981) have recently extended this model by allowing the linear air–sea coupling parameter to fluctuate. The growth rates for long-wavelength gravity waves predicted by the model of West & Seshadri exceed those predicted by the Miles–Phillips model by an order of magnitude for some wavelengths, in close agreement with field data.

In these linear-growth models the energy influx from the air flow stimulates an exponential growth of the water-wave amplitudes. In this paper we report on a dynamic model of the air–sea interaction which yields asymptotically a steady-state distribution of gravity–capillary waves. The nonlinear interactions among the waves are included in an 'optimum' linear equation and are shown to quench the wind-generated instabilities. The experiments of Plant & Wright (1977) indicate that 10 cm waves are probably the *longest* waves which grow by the direct influx of energy from the air flow, so these are the waves with which we will be concerned. The analysis is therefore restricted to fetches sufficiently long that the high-frequency water waves have time to reach their steady-state level, but short enough that the long-wavelength gravity waves have not yet attained appreciable amplitude.

No dynamic theory has been previously worked out which describes the evolution of the water surface driven by a fluctuating wind field from a state of rest to an asymptotic steady state. The dynamic models of Miles (1957, 1960), Phillips (1957) and West & Seshadri (1981) provide a description of the initial stages of wind-stimulated growth of water waves. In the gravity–capillary region of the spectrum the Miles–Phillips model does quite well in predicting the initial growth rates and is equivalent to the model of West & Seshadri. However, these models do not yield an asymptotic steady-state energy-spectral density. The steady state observed in the data is presumed to be a consequence of the nonlinear interactions among the water waves, see e.g., Kitaigorodskii (1973) or Phillips (1977) for a qualitative discussion of this effect. Herein we obtain a solution to the dynamic equations including the average effect of the nonlinear interaction and do indeed obtain such a steady-state energy-spectral density for the wind–wave field.

Van Dorn (1953) observed experimentally that there is a significant reduction of the momentum flux to the surface-wave field from the wind in the absence of short waves. Dobson (1971), in his measurements of the pressure perturbations induced in the air flow by low-frequency water waves, determined experimentally that a large fraction of the momentum flux from the air to the sea goes initially into the water-wave field rather than into a surface-drift current. Further, he observed that the rates of growth of these long waves exceeds those predicted by Miles' (1957) inviscid laminar theory of wave instability by a factor of between 5 and 8. Dobson resolved an apparent inconsistency between his data and those of Van Dorn by conjecturing that the high-frequency short waves act as a catalyst to the 'low-frequency' growth mechanism. We do not test that conjecture here, but we do assume that the dynamics of the short surface waves are very different from those of the long waves. We mention this mechanism here only because it is a process by which energy can be extracted from the high-frequency region of the spectrum, and it is not included in the present model. We determine that this mechanism may possibly explain an inconsistency noted between the calculated and experimental shape of the energy spectral density as a function of frequency.

We adopt the linear air–sea coupling models of Miles (1957, 1960) and Phillips (1957) to describe the initial growth of the high-frequency waves. The mean shear-flow instability in the air and the turbulent eddies in the wind field act in concert to generate the fluctuating water-wave field in this model. In addition to these two growth mechanisms, we include in the equations of motion viscous damping and the nonlinear interaction of the gravity–capillary waves among themselves. In § 2 we describe the dynamics by a system of nonlinear-mode-rate equations driven by a fluctuating inhomogeneity. In § 3 this system of *nonlinear stochastic differential equations* is replaced by a system of *linear stochastic differential equations* using the method of statistical linearization. The resulting linear system is a simple Langevin equation for the water-wave field for which a corresponding Fokker–Planck equation is constructed. These equations can be solved exactly, and yield a steady-state spectrum of gravity–

capillary waves in terms of the wind power spectral density and a *renormalized* Gaussian distribution for the water-wave statistics. In § 4 the statistical-linearization approximation is applied to a model nonlinear system to provide an interpretation of the technique in terms of renormalized perturbation theory. In § 5 we construct and solve formally the transport equation for the energy-spectral density of the gravity–capillary waves. In § 6 we calculate numerically the steady-state spectral density and relaxation rate of the high-frequency waves, which are seen to compare favourably with experiment.

2. Dynamic equation

In this paper we describe the fluid velocity $\mathbf{v}(\mathbf{x}, z, t)$ by means of a velocity potential $\phi(\mathbf{x}, z, t)$ located by the horizontal co-ordinate $\mathbf{x} = (x, y)$ and vertical displacement z. For a fluid with a pressure distribution $p(\mathbf{x}, z, t)$ acting on the free surface $z = \zeta(\mathbf{x}, t)$, the equations of motion are given by (see e.g. Landau & Lifshitz 1959)

$$\left.\begin{aligned}\frac{\partial\phi}{\partial t}+\tfrac{1}{2}\nabla\phi.\nabla\phi+g\zeta-\gamma\nabla_{\mathbf{x}}^2\zeta\{1+(\nabla_{\mathbf{x}}\zeta)^2\}^{-\frac{3}{2}} &= 2\nu\nabla_{\mathbf{x}}^2\phi-p/\rho_{\mathrm{w}},\\ \frac{\partial\zeta}{\partial t}+\nabla_{\mathbf{x}}\phi.\nabla_{\mathbf{x}}\zeta &= \frac{\partial\phi}{\partial z},\end{aligned}\right\} \tag{2.1}$$

defined on $z = \zeta(\mathbf{x}, t)$. In (2.1) γ is the kinematic surface tension, ν is the coefficient of kinematic viscosity, $\nabla_{\mathbf{x}}$ is the horizontal gradient operation $(\partial/\partial x, \partial/\partial y)$, and the constant fluid density is ρ_{w}. In this model of the fluid motion we neglect the rotational component of the fluid velocity that is generated by the viscous damping, i.e. the vorticity, and assume that the dominant characteristics of the surface motion can be described by irrotational flow alone. Finally, we assume that the fluid depth is much greater than the length scales of the surface motion, and that the surface is large in lateral extent. These latter two assumptions allow us to ignore the effects of the fixed boundaries on the motion of the surface and to separate completely the motion of the surface from that in the fluid interior.

Watson & West (1975) introduce the velocity potential at the surface by

$$\phi_{\mathrm{s}}(\mathbf{x}, t) \equiv \phi(\mathbf{x}, z, t) \quad \text{at} \quad z = \zeta(\mathbf{x}, t), \tag{2.2}$$

and represent the flow field in terms of wave modes by expanding both the velocity potential and the surface displacement in finite Fourier series:

$$\zeta(\mathbf{x}, t) = \sum_{\mathbf{k}} \zeta_{\mathbf{k}}(t)\, e^{i\mathbf{k}.\mathbf{x}}, \tag{2.3}$$

$$\phi_{\mathrm{s}}(\mathbf{x}, t) = \sum_{\mathbf{k}} \phi_{\mathbf{k}}(t)\, e^{i\mathbf{k}.\mathbf{x}}. \tag{2.4}$$

The Fourier amplitudes $\zeta_{\mathbf{k}}(t)$ and $\phi_{\mathbf{k}}(t)$ are identified with the time-dependent amplitudes of surface-wave modes of wave vector $\mathbf{k}$ defined on a rectangular area of ocean Σ_0 (with periodic boundary conditions). The Fourier exponentials satisfy the relations

$$\frac{1}{\Sigma_0}\int d^2x\, e^{i\mathbf{k}.\mathbf{x}} = \delta_{\mathbf{k},0}, \quad \frac{1}{\Sigma_0}\sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} = \delta(\mathbf{x}), \tag{2.5a, b}$$

where $\delta_{\mathbf{k},0}$ is the Kronecker delta and $\delta(\mathbf{x})$ the Dirac delta function on the two-dimensional quiescent ocean surface.

The coupling of the sea surface to the turbulent wind field in the Phillips–Miles model consists of the incoherent pressure fluctuations $p_0(\mathbf{x},t)$ and the in-phase pressure variations

$$p_1(\mathbf{x},t) = 2\rho_w \mu_\kappa V_\kappa \dot{\zeta}(\mathbf{x},t), \tag{2.6}$$

yielding the total-pressure field

$$p(\mathbf{x},t) = p_0(\mathbf{x},t) + p_1(\mathbf{x},t). \tag{2.7}$$

The operator V_κ weights each surface wave mode with the phase velocity of a small-amplitude gravity–capillary wave $V_k = (g/k+\gamma k)^{\frac{1}{2}}$. The operator μ_κ represents the fractional increase in the surface energy per radian. When applied to the Fourier transform of the surface field variable μ_κ yields the k-dependent quantity determined by Miles (1960).

In appendix A we express the equations of motion (2.1) in terms of quantities defined on the free surface. Using these expressions, we rewrite (2.1) in terms of the Fourier-mode amplitude as

$$\left[\frac{\partial}{\partial t} + 2(\nu k^2 - \mu_k \omega_k)\right]\phi_\mathbf{k} + (g+\gamma k^2)\,\zeta_\mathbf{k} = -p_\mathbf{k}(t)/\rho_w + F_\phi(\mathbf{k}), \tag{2.8a}$$

$$\frac{\partial}{\partial t}\zeta_\mathbf{k} - k\phi_\mathbf{k} = F_\zeta(\mathbf{k}), \tag{2.8b}$$

where the fluctuating air pressure at the free surface is given by the Fourier-series expansion

$$p_0(\mathbf{x},t) = \sum_\mathbf{k} p_\mathbf{k}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}},$$

and $p_\mathbf{k}(t) = p^*_{-\mathbf{k}}(t)$ since the pressure is a real quantity. Here ω_k is the angular frequency $(gk+\gamma k^3)$ and the functions $F_\phi(\mathbf{k},t)$ and $F_\zeta(\mathbf{k},t)$ are defined by the Fourier transforms of the nonlinear terms in (A 15a) and (A 15b), respectively. Note that the originally linear models of viscous dissipation and in-phase coupling to the mean air flow give rise to nonlinear terms at the free surface, i.e. nonlinear terms involving νk^2 and μ_k in the functions F_ϕ and F_ζ.

The nonlinear terms in (2.8) are treated here in a mode-coupled representation. To transform (2.8) to a system of normal-mode equations we diagonalize the deterministic linear part, i.e. omit $p_\mathbf{k}(t)$, $F_\phi(\mathbf{k})$ and $F_\zeta(\mathbf{k})$ from (2.8), and write the truncated equations in matrix form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi_\mathbf{k}(t)\\ \zeta_\mathbf{k}(t)\end{pmatrix} = -\begin{pmatrix}2(\nu k^2-\mu_k\omega_k) & (g+\gamma k^2)\\ -k & 0\end{pmatrix}\begin{pmatrix}\phi_\mathbf{k}(t)\\ \zeta_\mathbf{k}(t)\end{pmatrix}. \tag{2.9}$$

Equation (2.9) can be diagonalized to yield

$$\frac{\partial}{\partial t}\begin{pmatrix}B_\mathbf{k}(t)\\ B^*_\mathbf{k}(t)\end{pmatrix} + \begin{pmatrix}\lambda_k & 0\\ 0 & \lambda^*_k\end{pmatrix}\begin{pmatrix}B_\mathbf{k}(t)\\ B^*_\mathbf{k}(t)\end{pmatrix} = 0, \tag{2.10}$$

where λ_k is the eigenvalue given by

$$\lambda_k = \nu k^2 - \mu_k\omega_k + i\omega_1(k), \tag{2.11}$$

with

$$\omega_1(k) = [\omega_k^2 - (\nu k^2 - \mu_k\omega_k)^2]^{\frac{1}{2}}. \tag{2.12}$$

Here $\omega_1(k)$ is the linear frequency shifted by the effect of viscous dissipation νk^2 and the coupling to the air flow $\mu_k \omega_k$. The linear-eigenmode amplitudes in (2.10) are obtained from the transformation

$$\begin{pmatrix} B_{\mathbf{k}}(t) \\ B_{\mathbf{k}}^*(t) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\mathscr{V}_k} & -i \\ \dfrac{1}{\mathscr{V}_k^*} & i \end{pmatrix} \begin{pmatrix} \phi_{\mathbf{k}}(t) \\ \zeta_{\mathbf{k}}(t) \end{pmatrix}, \tag{2.13}$$

where the linear eigenvalues are used to define the 'complex velocity' $\mathscr{V}_k$, i.e.

$$\mathscr{V}_k \equiv -i\lambda_k/k, \tag{2.14}$$

which in the absence of wind ($\mu_k = 0$) and viscosity ($\nu = 0$) reduces to the ordinary phase velocity $V_k = (g/k + \gamma k)^{\frac{1}{2}}$ of a small-amplitude linear water wave.

By applying the transformation (2.13) to the system of nonlinear equations (2.8), we obtain the mode-rate equations

$$\dot{B}_{\mathbf{k}}(t) + \lambda_k B_{\mathbf{k}}(t) = f_{\mathbf{k}}(t) + T_{\mathbf{k}}(\mathbf{B}). \tag{2.15}$$

The function $T_{\mathbf{k}}(\mathbf{B})$ is the sum of the quadratic nonlinear wave–wave interactions

$$T_{\mathbf{k}}^{(2)}(\mathbf{B}) = \sum_{\mathbf{l},\mathbf{m}} \delta_{\mathbf{k}-\mathbf{l}-\mathbf{m}} \{C_{\mathbf{lm}}^{\mathbf{k}} B_{\mathbf{l}} B_{\mathbf{m}} + C_{\mathbf{l}}^{\mathbf{k},-\mathbf{m}} B_{\mathbf{l}} B_{-\mathbf{m}}^* + C^{\mathbf{k},-\mathbf{l},-\mathbf{m}} B_{-\mathbf{l}}^* B_{-\mathbf{m}}^*\}, \tag{2.16}$$

and the cubic nonlinear interactions

$$T_{\mathbf{k}}^{(3)}(\mathbf{B}) = \sum_{\mathbf{l},\mathbf{m},\mathbf{n}} \delta_{\mathbf{k}+\mathbf{n}-\mathbf{l}-\mathbf{m}} \{C_{\mathbf{lmn}}^{\mathbf{k}} B_{\mathbf{l}} B_{\mathbf{m}} B_{\mathbf{n}} + C_{\mathbf{lm}}^{\mathbf{kn}} B_{\mathbf{l}} B_{\mathbf{m}} B_{\mathbf{n}}^* + C_{\mathbf{l}}^{\mathbf{kmn}} B_{\mathbf{l}} B_{\mathbf{m}}^* B_{\mathbf{n}}^* + C^{\mathbf{klmn}} B_{\mathbf{l}}^* B_{\mathbf{m}}^* B_{\mathbf{n}}^*\}. \tag{2.17}$$

The coupling coefficients in the quadratic terms are complex owing to the dependence on the coefficient of viscosity and the air–sea coupling parameter, and therefore differ from those obtained in other studies; see e.g. Valenzuela & Laing (1972) or Holliday (1977). The coefficients in (2.16) and (2.17) are recorded in appendix B. Only the coupling coefficient $C_{\mathbf{lm}}^{\mathbf{kn}}$ from the cubic terms in (2.17) is listed in appendix B, since this term is found subsequently to have the dominant effect on the equilibration of the gravity–capillary waves. The function $f_{\mathbf{k}}(t)$ models the turbulent eddies in the wind field and is a fluctuating function of time, i.e.

$$f_{\mathbf{k}}(t) \equiv i \frac{p_{\mathbf{k}}(t)}{\rho_w \mathscr{V}_k}, \tag{2.18}$$

where $p_{\mathbf{k}}(t)$ is the Fourier transform of the stochastic component of the pressure field at the ocean surface.

The expression (2.15) is therefore a *differential equation* not unlike those that arise in the study of turbulence; see e.g. Leslie (1973) for a review of such models. The formal similarity of (2.15) to expressions that have been studied, for example, by Kraichnan (1975), suggests that this wave field has a number of properties in common with those of turbulent fluid flow. The scaling argument used by Kitaigorodskii (1962) and also by Phillips (1977) are based in part on such an analogy. This correspondence cannot be pushed too far, however, because of the existence of a dispersion relation for the water waves, which is not present in turbulent flows. In addition there is the restriction that the dominant nonlinear interaction among gravity–capillary waves arises from *resonant triads* of waves (see e.g. Phillips 1960; Benney 1962; Longuet-Higgins 1962), satisfying the conditions on wave vectors and frequencies:

$$\mathbf{k} = \mathbf{l}+\mathbf{m}, \quad \omega_k = \omega_l + \omega_m. \tag{2.19}$$

In the following sections we examine the physical properties of an approximate solution to (2.15), subject to the restriction (2.19) on the nonlinear interactions.

The properties of the solution to the stochastic equation (2.15) can only be obtained when the statistics of the fluctuating force (2.18) are specified. The statistical properties of the solution are determined by averaging powers of the mode amplitudes over the fluctuations in the wind field. To define this average we ignore processes such as the generation of a mean drift current at the fluid surface (see e.g. Banner & Phillips 1974) and assume that the average energy influx from the wind is adequately modelled by the in-phase air–sea coupling term (2.6). This enables us to treat the turbulent fluctuation in the wind field as zero-centred, since the mean effect is included in (2.6). As mentioned previously, in the field measurements of low-frequency waves by Dobson (1971) it was observed that most of the momentum flux goes directly into the surface-wave field, so that this assumption is not unreasonable.

The solution to (2.15) is different for each realization of the air flow $f_{\mathbf{k}}(t)$. To determine the statistics of the solution we use a standard procedure from statistical mechanics and collect a large number of realizations of $f_{\mathbf{k}}(t)$ into an ensemble. It is then customary in obtaining average properties to replace all time averages by averages over the ensemble of realizations of $f_{\mathbf{k}}(t)$. Such an average we denote by a bracket with a f subscript. The first statistical property of the atmosphere at the air–sea interface is therefore given by

$$\langle f_{\mathbf{k}}(t)\rangle_f = 0. \tag{2.20a}$$

For the second-order statistics of $\mathbf{f}(t)$, noticing that the pressure field may be correlated over time intervals longer than the characteristic time scales of the high-frequency surface-wave spectrum, using (2.18) we write

$$\langle f_{\mathbf{k}}(t) f^*_{\mathbf{k}'}(t-\tau)\rangle_f = \frac{2}{\rho_{\mathrm{w}}^2 \mathscr{V}_k \mathscr{V}^*_{k'}} \langle p_{\mathbf{k}}(t) p^*_{\mathbf{k}'}(t-\tau)\rangle_f.$$

Assuming a spatially homogeneous and temporally stationary pressure field, we obtain

$$\langle f_{\mathbf{k}}(t) f^*_{\mathbf{k}'}(t-\tau)\rangle_f = \frac{\delta_{\mathbf{k}-\mathbf{k}'}}{\rho_{\mathrm{w}}^2 |\mathscr{V}_k|^2} \Phi(\mathbf{k},\tau). \tag{2.20b}$$

The strength of the correlations in the air flow on the scale $2\pi/k$ over a time interval τ is given by $\Phi(\mathbf{k},\tau)$. The time Fourier transform of the correlation function, i.e. $\tilde{\Phi}(\mathbf{k},\omega)$, is the three-dimensional space–time power-spectral density of the pressure field. Because the complex function $f_{\mathbf{k}}(t)$ is proportional to the Fourier transform of a real stochastic function, i.e. $p_{\mathbf{k}}(t) = p^*_{-\mathbf{k}}(t)$, we also have

$$\langle f_{\mathbf{k}}(t) f_{\mathbf{k}'}(t-\tau)\rangle_f = -\frac{\delta_{\mathbf{k}+\mathbf{k}'}}{\rho_{\mathrm{w}}^2 \mathscr{V}_k^2} \Phi(\mathbf{k},\tau) = \langle f_{\mathbf{k}}(t) f_{\mathbf{k}}(t-\tau)\rangle_f = 0. \tag{2.20c}$$

In addition we assume that all higher cumulants of $p_{\mathbf{k}}(t)$ vanish. These statistical properties of the surface-pressure field specify that $f_{\mathbf{k}}(t)$ is a zero-centred, homogeneous, stationary Gaussian random process with power-spectral density $\tilde{\Phi}(\mathbf{k},\omega)$ and a complex coefficient.

Given these statistical properties of $f_{\mathbf{k}}(t)$, the analytic solution to (2.15) is still beyond our grasp because of the nonlinear interaction function $T_{\mathbf{k}}$ in the equation of motion. We therefore turn to an approximation scheme called statistical linearization, which has been developed in the engineering community (see e.g. Caughey 1963; Crandall 1973).

3. Statistical linearization

The technical difficulties associated with the mathematical and physical analysis of nonlinear stochastic differential equations such as (2.15) are well known from studies in turbulence (Leslie 1973), analytical dynamics (Ford 1961; Chirikov 1978; Tabor 1981), statistical mechanics (Zwanzig 1972; Kawasaki 1970) and many other areas of physics and applied mathematics (see e.g. Lax 1966; Van Kampen 1976). To circumvent these difficulties we intend to replace the nonlinear system of gravity–capillary waves by an *optimum* linear system in which the nonlinearities enter through a self-consistent renormalization of the frequency and growth rate. The approximation techniques, unlike ordinary perturbation theory, which is valid at early times, are valid at late times. This asymptotic technique involves replacing the nonlinear function $T_{\mathbf{k}}(\mathbf{B})$ with a 'statistically equivalent' function no higher than linear in the mode amplitude $B_{\mathbf{k}}(t)$. The method is referred to as statistical linearization and has been favourably compared with high-order perturbation theories by West, Lindenberg & Shuler (1978) and also by Budgor & West (1978). The method reproduces the steady-state average of $B_{\mathbf{k}}(t)$ and yields a self-consistent expression for the steady-state moments. Although the first two moments are certainly an inadequate description of the distribution function for the system (unless it just happens to be Gaussian), they can provide a good estimate of the steady-state second-order statistics, i.e. variances, correlation functions and spectral densities (see e.g. Budgor, Lindenberg & Shuler (1976), and for an extension to systems with many degrees of freedom see West *et al.* (1978)). These are precisely the properties which interest us here, so the approximation is felt to be adequate.

The prescription we use in this approximation is to replace the nonlinear function $T_{\mathbf{k}}(\mathbf{B})$ by a term no higher than linear in $B_{\mathbf{k}}(t)$, i.e.

$$T_{\mathbf{k}}(B) \rightarrow \beta_{\mathbf{k}} + h_{\mathbf{k}} B_{\mathbf{k}}(t), \tag{3.1}$$

where $\beta_{\mathbf{k}}$ and $h_{\mathbf{k}}$ are complex parameters selected so that the mean-square error $\epsilon_{\mathbf{k}}^2$ due to this replacement,

$$\epsilon_{\mathbf{k}}^2 \equiv \lim_{T\rightarrow\infty} \frac{1}{2T}\int_{-T}^{T} |T_{\mathbf{k}}(\mathbf{B},t) - \beta_{\mathbf{k}} - h_{\mathbf{k}} B_{\mathbf{k}}(t)|^2 \, dt, \tag{3.2}$$

is a minimum. The initial conditions in (3.2) have been shifted to $t = -\infty$. The error-minimization condition is contained in the expressions for the independent variations of $\epsilon_{\mathbf{k}}^2$ with respect to $h_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$, i.e.

$$\frac{\delta\epsilon_{\mathbf{k}}^2}{\delta h_{\mathbf{k}}} = 0, \quad \frac{\delta\epsilon_{\mathbf{k}}^2}{\delta\beta_{\mathbf{k}}} = 0, \tag{3.3}$$

which from (3.2) yield

$$h_{\mathbf{k}} = \frac{\langle \hat{B}_{\mathbf{k}}^*(t)\, T_{\mathbf{k}}(\mathbf{B})\rangle_{\mathrm{T}}}{\langle |\hat{B}_{\mathbf{k}}(t)|^2\rangle_{\mathrm{T}}}, \tag{3.4}$$

$$\beta_{\mathbf{k}} = \langle T_{\mathbf{k}}(\mathbf{B})\rangle_{\mathrm{T}} - \langle B_{\mathbf{k}}(t)\rangle_{\mathrm{T}} \frac{\langle \hat{B}_{\mathbf{k}}^*(t)\, T_{\mathbf{k}}(\mathbf{B})\rangle_{\mathrm{T}}}{\langle |\hat{B}_{\mathbf{k}}(t)|^2\rangle_{\mathrm{T}}}. \tag{3.5}$$

The quantity $\hat{B}_{\mathbf{k}}$ ($\equiv B_{\mathbf{k}} - \langle B_{\mathbf{k}} \rangle_{\mathrm{T}}$) is the fluctuating component of the mode amplitude away from its time-averaged value, where the time average is indicated by the T subscript on the averaging brackets.

The evaluation of the parameter sets $\{\beta_{\mathbf{k}}\}$ and $\{h_{\mathbf{k}}\}$ requires a knowledge of the time-dependent solution of (2.15) in order to perform the indicated time averages in (3.4) and (3.5). The motivation for introducing this approximation procedure, however, is precisely that we do not know how to obtain an exact solution to (2.15). To skirt this problem in the calculation of $\beta_{\mathbf{k}}$ and $h_{\mathbf{k}}$, it is necessary to replace these time averages by a *steady-state* ensemble average, i.e. an average over an ensemble distribution that is valid at late time. This is a valid procedure if the system is ergodic. To obtain the steady-state distribution of the gravity–capillary wave field we solve the equation of evolution for the probability density in the phase space of the water-wave system. This equation of evolution when the flux $f_{\mathbf{k}}(t)$ is a Gaussian, delta-correlated process is a Fokker–Planck equation. The Fokker–Planck equation for the linearized gravity–capillary wave field can be solved exactly to obtain the steady-state distribution function with a parametric dependence on the quantities given by (3.4) and (3.5). Replacing the time averages by steady-state ensemble averages in these relations establishes a self-consistency requirement on the parameters $h_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$.

Using (3.1) in (2.15) we obtain the linearized equations of motion

$$\frac{d}{dt} B_{\mathbf{k}}(t) + \alpha(\mathbf{k}) B_{\mathbf{k}}(t) = \beta_{\mathbf{k}} + f_{\mathbf{k}}(t), \tag{3.6}$$

where the complex coefficient $\alpha(\mathbf{k}) \equiv \alpha_{\mathrm{R}}(\mathbf{k}) + i\alpha_{\mathrm{I}}(\mathbf{k})$ consists of the shifted frequency

$$\alpha_{\mathrm{I}}(\mathbf{k}) \equiv \mathscr{I}[\alpha(\mathbf{k})] = \omega_1(\mathbf{k}) - \mathscr{I}[h_{\mathbf{k}}], \tag{3.7}$$

and shifted 'growth parameter'

$$\alpha_{\mathrm{R}}(\mathbf{k}) \equiv \mathscr{R}[\alpha(\mathbf{k})] = \nu k^2 - \mu_k \omega_k - \mathscr{R}[h_{\mathbf{k}}]. \tag{3.8}$$

The first important property of (3.6) is that it preserves the evolution of the average mode amplitude. Taking the steady-state ensemble average of (3.6) and using the ergodic theorem to obtain the new definitions of $\beta_{\mathbf{k}}$ and $h_{\mathbf{k}}$,

$$h_{\mathbf{k}} = \langle \hat{B}^*_{\mathbf{k}}(t)\, T_{\mathbf{k}}(\mathbf{B}) \rangle_{\mathrm{ss}} / \langle |\hat{B}_{\mathbf{k}}(t)|^2 \rangle_{\mathrm{ss}}, \tag{3.9}$$

$$\beta_{\mathbf{k}} = \langle T_{\mathbf{k}}(\mathbf{B}) \rangle_{\mathrm{ss}} - h_{\mathbf{k}} \langle B_{\mathbf{k}}(t) \rangle_{\mathrm{ss}}, \tag{3.10}$$

it is immediately shown that

$$\frac{d}{dt} \langle B_{\mathbf{k}} \rangle_{\mathrm{ss}} + \lambda_k \langle B_{\mathbf{k}} \rangle_{\mathrm{ss}} = \langle T_{\mathbf{k}} \rangle_{\mathrm{ss}}, \tag{3.11}$$

which can be obtained directly from (2.15) by averaging over the steady-state distribution. The steady-state average in (3.9) and (3.10), or correspondingly the $T \to \infty$ limit in (3.4) and (3.5), implies that the variational parameters $h_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ are only valid in the asymptotic region. This is precisely the region where the gravity–capillary waves have received sufficient energy from the wind to make the nonlinear terms in $T_{\mathbf{k}}(\mathbf{B})$ non-negligible.

Using the definition of $\beta_{\mathbf{k}}$ given in (3.10) and the average equation (3.11), we can rewrite the linear equation of motion (3.6) in terms of the variation of the mode amplitude from its steady state value

$$\hat{B}_{\mathbf{k}}(t) \equiv B_{\mathbf{k}}(t) - \langle B_{\mathbf{k}}(t) \rangle_{\mathrm{ss}}, \tag{3.12}$$

as

$$\frac{d}{dt}\hat{B}_{\mathbf{k}}(t) + \alpha(\mathbf{k})\,\hat{B}_{\mathbf{k}}(t) = f_{\mathbf{k}}(t). \tag{3.13}$$

Equation (3.13) is a linear Langevin equation for the complex mode amplitude $\hat{B}_{\mathbf{k}}(t)$, and includes *all* the dominant effects in our model of the gravity–capillary wave field. Through λ_k this equation includes viscous dissipation and the resonant influx of energy from the mean wind field; through $f_{\mathbf{k}}(t)$ it includes the incoherent influx of energy from the fluctuations in the air flow, and finally through the parameter $h_{\mathbf{k}}$ it includes the average nonlinear interactions among the waves at late times. The construction of the linear equation (3.13) has been deceptively easy and it is important to understand the physical content of the variational procedure (3.3) in order to appreciate the degree of validity of this approximation technique. We examine this interpretation in § 4.

Before we concentrate on the interpretation of the above result we note that all physical quantities are related to an average over the steady-state distribution function; a quantity we do not yet have. However, the final equation of evolution is a linear Langevin equation, so that we can construct the steady-state distribution function by solving the phase-space equation of evolution for the probability

$$P(\mathbf{b}, t \mid \mathbf{b}_0)\, d\Gamma(\mathbf{b})$$

that the dynamic variables $\hat{\mathbf{B}}(t)$ has a value in the interval $(\mathbf{b}, \mathbf{b} + d\mathbf{b})$ at time t given an initial value $\hat{\mathbf{B}}(t=0) = \mathbf{b}_0$. Here $d\Gamma(\mathbf{b})$ is a differential volume element in the $2N$-dimensional phase space for the system of N gravity–capillary waves. Using (3.13), we define the operators L_0 and $L_f(t)$ such that if $G(\mathbf{b})$ is an arbitrary phase-space function, then

$$L_0 G(\mathbf{b}) \equiv \sum_{\mathbf{k}} \frac{\partial}{\partial b_{\mathbf{k}}} [\alpha(\mathbf{k})\, b_{\mathbf{k}}\, G(\mathbf{b})] + \text{c.c.} \tag{3.14a}$$

$$L_f(t)\, G(\mathbf{b}) \equiv -\sum_{\mathbf{k}} \frac{\partial}{\partial b_{\mathbf{k}}} [f_{\mathbf{k}}(t)\, G(\mathbf{b})] + \text{c.c.}\,. \tag{3.14b}$$

It has been shown by a number of investigators (e.g. Van Kampen 1976; Mukamel, Oppenheim & Ross 1978; West *et al.* 1979) that when $f_{\mathbf{k}}(t)$ has zero-centred Gaussian statistics, the *exact* equation of evolution for $P(\mathbf{b}, t \mid \mathbf{b}_0)$ is given by

$$\frac{\partial}{\partial t} P(\mathbf{b}, t \mid \mathbf{b}_0) = L_0 P(\mathbf{b}, t \mid \mathbf{b}_0) + \int_0^t \langle L_f(t)\, e^{L_0 \tau} L_f(t-\tau)\, e^{-L_0 \tau} \rangle\, P(\mathbf{b}, t \mid \mathbf{b}_0)\, d\tau. \tag{3.15}$$

Introducing the derivative operator in the interaction representation

$$\frac{\partial}{\partial \overline{b}_{\mathbf{k}}(t)} \equiv e^{L_0 t} \frac{\partial}{\partial b_{\mathbf{k}}} e^{-L_0 t} \tag{3.16}$$

(see e.g. Lax 1966; West *et al.* (1979), and using the second-order statistics of $f_{\mathbf{k}}(t)$ given by (2.20), (3.15) assumes the form

$$\frac{\partial P}{\partial t}(\mathbf{b},t\,|\,\mathbf{b}_0) = L_0 P(\mathbf{b},t\,|\,\mathbf{b}_0) + \sum_{\mathbf{k},\mathbf{k}'} \int_0^t d\tau \left\{ \frac{\partial}{\partial b_{\mathbf{k}}} \left[\left\langle f_{\mathbf{k}}(t) f_{\mathbf{k}'}(t-\tau) \frac{\partial}{\partial \bar{b}_{\mathbf{k}'}(\tau)} \right\rangle \right.\right.$$
$$\left. + \left\langle f_{\mathbf{k}}(t) f^*_{\mathbf{k}'}(t-\tau) \frac{\partial}{\partial \bar{b}^*_{\mathbf{k}'}(\tau)} \right\rangle \right] + \frac{\partial}{\partial b^*_{\mathbf{k}}} \left[\left\langle f^*_{\mathbf{k}}(t) f_{\mathbf{k}'}(t-\tau) \frac{\partial}{\partial \bar{b}_{\mathbf{k}'}(\tau)} \right\rangle \right.$$
$$\left.\left. + \left\langle f^*_{\mathbf{k}}(t) f_{\mathbf{k}'}(t-\tau) \frac{\partial}{\partial \bar{b}^*_{\mathbf{k}'}(\tau)} \right\rangle \right] \right\} P(\mathbf{b},t\,|\,\mathbf{b}_0). \tag{3.17}$$

Factoring the averages in (3.17) using (2.30), and introducing

$$\Phi_{\mathbf{k}}(\tau) \equiv \frac{k^2 \Phi(\mathbf{k},\tau)}{\rho_{\mathrm{w}}^2}, \tag{3.18}$$

we can write

$$\frac{\partial P(\mathbf{b},t\,|\,\mathbf{b}_0)}{\partial t} = L_0 P(\mathbf{b},t\,|\,\mathbf{b}_0) + \sum_{\mathbf{k}} \int_0^t d\tau \Phi_{\mathbf{k}}(\tau) \left\{ \left(\frac{\partial b_{-\mathbf{k}}}{\partial \bar{b}_{-\mathbf{k}}(\tau)} \right) \frac{\partial^2}{\partial(\lambda_k b_{\mathbf{k}})\,\partial(\lambda_{-k} b_{-\mathbf{k}})} \right.$$
$$\left. + \left(\frac{\partial b_{-\mathbf{k}}}{\partial \bar{b}_{-\mathbf{k}}(\tau)} \right)^* \frac{\partial^2}{\partial(\lambda^*_{\mathbf{k}}\, b^*_{\mathbf{k}})\,\partial(\lambda^*_{-k} b^*_{-\mathbf{k}})} + \left[\left(\frac{\partial b_{\mathbf{k}}}{\partial \bar{b}_{\mathbf{k}}(\tau)} \right)^* + \left(\frac{\partial b_{\mathbf{k}}}{\partial \bar{b}_{\mathbf{k}}(\tau)} \right) \right] \frac{\partial^2}{(\partial \lambda_k b_{\mathbf{k}})\,\partial(\lambda^*_k b^*_{\mathbf{k}})} \right\}$$
$$\times P(\mathbf{b},t\,|\,\mathbf{b}_0), \tag{3.19}$$

where $\partial b_{\mathbf{k}}/\partial \bar{b}_{\mathbf{k}}(\tau)$ is the Jacobian of the deterministic time evolution of the mode $\hat{B}_{\mathbf{k}}(t)$ and in the linearized system (3.13) is given by

$$\left(\frac{\partial b_{\mathbf{k}}}{\partial \bar{b}_{\mathbf{k}}(\tau)} \right) = e^{-\alpha(\mathbf{k})\tau}. \tag{3.20}$$

Therefore, assuming time-reversal symmetry in the autocorrection function of the pressure-field fluctuations, i.e. $\Phi_{\mathbf{k}}(\tau) = \Phi_{\mathbf{k}}(-\tau)$, introducing the function

$$\tilde{\Phi}_{\mathrm{c}}[\alpha(\mathbf{k})] = \int_0^\infty d\tau\, \Phi_{\mathbf{k}}(\tau)\, e^{-\alpha_{\mathrm{R}}(\mathbf{k})\tau} \cos[\alpha_{\mathrm{I}}(\mathbf{k})\,\tau] = \frac{k^2}{\rho_{\mathrm{w}}^2} \tilde{\Phi}_{\mathrm{c}}[\mathbf{k},\alpha(\mathbf{k})], \tag{3.21}$$

and assuming t in (3.19) to be very much greater than the correlation time τ_{c} of the pressure field fluctuations, we replace the upper limit of the integral by infinity. In order to do this we are implicitly assuming that $\alpha_{\mathrm{R}} > 0$, a condition we prove later by direct calculation. Thus, restricting our considerations to waves propagating in the direction of the wind only (3.19) reduces to

$$\frac{\partial P(\mathbf{b},t\,|\,\mathbf{b}_0)}{\partial t} = \sum_{\mathbf{k}} \left\{ \frac{\partial}{\partial b_{\mathbf{k}}} [\alpha(\mathbf{k})\, b_{\mathbf{k}}] + \text{c.c.} + 2D_{\mathbf{k}} \frac{\partial^2}{\partial b_{\mathbf{k}}\, \partial b^*_{\mathbf{k}}} \right\} P(\mathbf{b},t\,|\,\mathbf{b}_0), \tag{3.22}$$

where

$$D_{\mathbf{k}} = \frac{\tilde{\Phi}_{\mathrm{c}}[\mathbf{k},\alpha(\mathbf{k})]}{\rho_{\mathrm{w}}^2\, V_k^2}. \tag{3.23}$$

The steady-state solution to (3.22) is obtained from

$$\frac{\partial P_{\mathrm{ss}}(\mathbf{b})}{\partial t} = 0 \tag{3.24}$$

to be

$$P_{\mathrm{ss}}(\mathbf{b}) = \frac{\alpha_{\mathrm{R}}(\mathbf{k})}{\pi D_{\mathbf{k}}} \exp\left\{ -\frac{\alpha_{\mathrm{R}}(\mathbf{k})}{D_{\mathbf{k}}} b_{\mathbf{k}} b^*_{\mathbf{k}} \right\}, \tag{3.25}$$

which is a Gaussian distribution in the complex mode amplitude $b_{\mathbf{k}}$. It is important to distinguish the use of (3.25) from the quasi-Gaussian approximation that one usually encounters in the study of water-wave interactions. The first distinction is that (3.25) depends parametrically on the average nonlinear interaction through $\alpha(\mathbf{k}) = \lambda_k - h_{\mathbf{k}}$. Thus, when we evaluate the steady-state energy-spectral density $F_{ss}(\mathbf{k})$ using (3.25), i.e.

$$F_{ss}(\mathbf{k}) = \tfrac{1}{2}\langle|\hat{B}_{\mathbf{k}}(t)|^2\rangle_{ss} = \tfrac{1}{2}\int |b_{\mathbf{k}}|^2 P_{ss}(\mathbf{b})\, d\Gamma(\mathbf{b}), \tag{3.26}$$

we obtain an explicit dependence on the average nonlinear interactions, i.e.

$$F_{ss}(\mathbf{k}) = \frac{D_{\mathbf{k}}}{2\alpha_{\mathrm{R}}(\mathbf{k})} = \frac{\tilde{\Phi}_{\mathrm{c}}[\mathbf{k}, \alpha(\mathbf{k})]}{2\rho_{\mathrm{w}}^2 V_k^2[\lambda_k - \mathscr{R}h_{\mathbf{k}}]}. \tag{3.27}$$

For a prescribed spectrum of pressure-field fluctuations $\tilde{\Phi}_{\mathrm{c}}$ from (3.21) and a given $h_{\mathbf{k}}$ we can then *calculate* the steady-state energy-spectral density of the gravity–capillary field. The second distinction is that the parameter $h_{\mathbf{k}}$ must be calculated self-consistently with the distribution (3.25), i.e. using (3.9) we obtain

$$h_{\mathbf{k}} = \frac{\int d\Gamma(\mathbf{b})\, \mathbf{b}_{\mathbf{k}}^* T_{\mathbf{k}}(\mathbf{b})\, P_{ss}(\mathbf{b})}{\int d\Gamma(\mathbf{b})\, |b_{\mathbf{k}}|^2 P_{ss}(\mathbf{b})}, \tag{3.28}$$

yielding an expression with $h_{\mathbf{k}}$ on both the left- and right-hand sides of the equation, i.e.

$$\begin{aligned} h_{\mathbf{k}} &= 2\sum_{\mathbf{l}} \bar{C}_{\mathbf{kl}}^{\mathbf{kl}} \langle|\mathbf{b}_{\mathbf{l}}|^2\rangle_{ss} \\ &= 4\sum_{\mathbf{l}} \bar{C}_{\mathbf{kl}}^{\mathbf{kl}} F_{ss}(\mathbf{l}), \end{aligned} \tag{3.29}$$

since $F_{ss}(\mathbf{l})$ as given by (3.27) depends on $h_{\mathbf{l}}$. The coupling coefficient $\bar{C}_{\mathbf{kl}}^{\mathbf{kl}}$ is discussed in appendix B, and involves virtual four-wave interactions induced by the three-wave interaction terms as well as the direct four-wave interactions.

The expressions (3.27) and (3.29) taken together can be interpreted as an asymptotic renormalization of the linear eigenvalues produced by the nonlinear wave–wave interactions. A more transparent example of this effect is provided in § 4 to stress the physical interpretation of the statistical-linearization approximation.

4. Interpretation of statistical linearization

The comparison of the method of statistical linearization with perturbation and projection-operator techniques has been made by West *et al.* (1978), Budgor & West (1978) and West (1980). We present some of that discussion in this section to clarify the interpretation of the variational prescription (3.3) in terms of more familiar techniques. In particular, we focus our discussion on the perturbation solution of a system of mode-rate equations

$$\dot{a}_{\mathbf{k}}(t) + (i\omega_k + \gamma_k)\, a_{\mathbf{k}}(t) + \Gamma_{\mathbf{k}} \sum_{\{\mathbf{k}\}} a_{\mathbf{k}_1}(t)\, a_{\mathbf{k}_2}(t)\, a_{\mathbf{k}_3}^*(t) = g_{\mathbf{k}}(t), \tag{4.1}$$

where, although for the purpose of this section it is not necessary to have a physical interpretation of (4.1), it might be thought of as describing the interaction among the waves in a narrow band spectrum of gravity waves, but with a complex $\Gamma_{\mathbf{k}}$. We are not interested in obtaining the general perturbation solution to (4.1) here, but rather in determining which terms in the perturbation solution correspond to the statistical linearization results. First, following West *et al.* (1978), we note that the statistically linearized equation corresponding to (4.1) is

$$\dot{a}_{\mathbf{k}}(t) + (i\omega_k + \gamma_k - h_{\mathbf{k}})\, a_{\mathbf{k}}(t) = g_{\mathbf{k}}(t), \tag{4.2}$$

with

$$h_{\mathbf{k}} = \Gamma_{\mathbf{k}} \frac{\sum_{\{\mathbf{k}\}} \langle a_{\mathbf{k}}^* a_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}^* \rangle_{\text{ss}}}{\langle |a_{\mathbf{k}}|^2 \rangle_{\text{ss}}}, \tag{4.3}$$

where the brackets denote an average over the self-consistent steady-state distribution function, and the $\{\mathbf{k}\}$ on the summation indicates a wave-vector matching condition $\mathbf{k} + \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$.

We begin the analysis by Fourier-transforming the mode amplitudes $a_{\mathbf{k}}(t)$ in time according to

$$\hat{a}_{\mathbf{k}\omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, a_{\mathbf{k}}(t)\, e^{-i\omega t}, \tag{4.4}$$

so that the nonlinear mode-rate equations (4.1) become

$$Q_{\mathbf{k}}^{-1}(\omega)\, \hat{a}_{\mathbf{k}\omega} = \Gamma_{\mathbf{k}} \sum_{\{\mathbf{k},\omega\}} \hat{a}_{\mathbf{k}_1\omega_1} \hat{a}_{\mathbf{k}_2\omega_2} \hat{a}_{\mathbf{k}_3\omega_3}^* + \hat{g}_{\mathbf{k}\omega}, \tag{4.5}$$

where

$$Q_{\mathbf{k}}^{-1}(\omega) \equiv \gamma_k + i(\omega_k - \omega), \tag{4.6}$$

and the noise spectrum is given by $|\hat{g}_{\mathbf{k}\omega}|^2$, where

$$\hat{g}_{\mathbf{k}\omega} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, g_{\mathbf{k}}(t)\, e^{-i\omega t}, \tag{4.7}$$

and for convenience we choose $g_{\mathbf{k}}$ to be a zero-centred, delta-correlated Gaussian process. The sum in (4.5) is over $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and $\omega_1, \omega_2, \omega_3$; there is a four-wave resonance condition given by

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}, \quad \omega_1 + \omega_2 = \omega_3 + \omega, \tag{4.8}$$

and the summations are replaced by integrals for continuous wavenumbers and frequencies. Note that there is no dispersion relation between the frequencies and wave vectors in (4.5) or (4.8).

Introducing the zeroth-order mode amplitude by

$$\hat{a}_{\mathbf{k}\omega}^{(0)} = Q_{\mathbf{k}}(\omega)\, \hat{g}_{\mathbf{k}\omega}, \tag{4.9}$$

the integral equation that we must solve is

$$\hat{a}_{\mathbf{k}\omega} = \hat{a}_{\mathbf{k}\omega}^{(0)} + \Gamma_{\mathbf{k}} Q_{\mathbf{k}}(\omega) \sum_{\{\mathbf{k},\omega\}} a_{\mathbf{k}_1\omega_1} a_{\mathbf{k}_2\omega_2} a_{\mathbf{k}_3\omega_3}^*. \tag{4.10}$$

For compactness we introduce the composite variable $\xi \equiv (\mathbf{k}, \omega)$ and define the propagator

$$S(\xi) \equiv \frac{\Gamma_{\mathbf{k}}}{\Gamma} Q(\xi), \tag{4.11}$$

where Γ can be interpreted as the average scale of $\Gamma_{\mathbf{k}}$. The actual value of Γ need not be known for the purposes here, but should be small, i.e. $\Gamma \ll 1$. The parameter Γ will be used as an expansion parameter in the perturbation series, and for convenience will be taken to be real. Equation (4.10) can now be rewritten as

$$\hat{a}(\xi) = \hat{a}^{(0)}(\xi) + \Gamma S(\xi) \sum_{\{\xi\}} \hat{a}(\xi_1)\, \hat{a}(\xi_2)\, \hat{a}^*(\xi_3), \tag{4.12}$$

so that introducing the series expansion

$$\hat{a}(\xi) = \hat{a}^{(0)}(\xi) + \hat{a}^{(1)}(\xi) + \hat{a}^{(2)}(\xi) + \dots \tag{4.13}$$

into (4.12) and equating like orders of Γ ($\hat{a}^{(m)}(\xi)$ is implicitly $O(\Gamma^m)$) we obtain the hierarchy of equations

$$\left.\begin{aligned}
\hat{a}^{(0)}(\xi) &= Q(\xi)\hat{g}(\xi),\\
\hat{a}^{(1)}(\xi) &= \Gamma S(\xi)\sum_{\{\xi\}}\hat{a}^{(0)}(\xi_1)\,\hat{a}^{(0)}(\xi_2)\,\hat{a}^{(0)*}(\xi_3),\\
\hat{a}^{(2)}(\xi) &= \Gamma S(\xi)\sum_{\{\xi\}}[2\hat{a}^{(0)}(\xi_1)\,\hat{a}^{(1)}(\xi_2)\,\hat{a}^{(0)*}(\xi_3) + \hat{a}^{(0)}(\xi_1)\,\hat{a}^{(0)}(\xi_2)\,\hat{a}^{(1)*}(\xi_3)],\\
\hat{a}^{(3)}(\xi) &= \Gamma S(\xi)\sum_{\{\xi\}}\{2[\hat{a}^{(0)}(\xi_1)\,\hat{a}^{(2)}(\xi_2)\,\hat{a}^{(0)*}(\xi_3) + \hat{a}^{(0)}(\xi_1)\,\hat{a}^{(1)}(\xi_2)\,\hat{a}^{(1)*}(\xi_3)]\\
&\quad + \hat{a}^{(0)}(\xi_1)\,\hat{a}^{(0)}(\xi_2)\,\hat{a}^{(2)*}(\xi_3) + \hat{a}^{(1)}(\xi_1)\,\hat{a}^{(1)}(\xi_2)\,\hat{a}^{(1)*}(\xi_3)\},\\
&\ \ \vdots \qquad\qquad \vdots
\end{aligned}\right\} \tag{4.14}$$

At each order in the hierarchy given by (4.14) one can insert the amplitudes from the preceding order, thereby obtaining expressions solely in terms of $\hat{a}^{(0)}(\xi)$, the propagators $S(\xi)$, and the coupling coefficient Γ. When this is done all, terms in $\hat{a}^{(m)}(\xi)$ are explicitly $O(\Gamma^m)$. Inserting the expressions from (4.14) into (4.12) and writing only the terms to $O(\Gamma^2)$ yields

$$\begin{aligned}
\hat{a}(\xi) &= \hat{a}^{(0)}(\xi) + \Gamma S(\xi)\sum_{\{\xi\}}\hat{a}^{(0)}(\xi_1)\,\hat{a}^{(0)}(\xi_2)\,\hat{a}^{(0)*}(\xi_3)\\
&\quad + \Gamma S(\xi)\sum_{\{\xi\}}2\hat{a}^{(0)}(\xi_1)\,\hat{a}^{(0)*}(\xi_3)\,\Gamma S(\xi_2)\sum_{\{\xi_2\}}\hat{a}^{(0)}(\xi_4)\,\hat{a}^{(0)}(\xi_5)\,\hat{a}^{(0)*}(\xi_6)\\
&\quad + \hat{a}^{(0)}(\xi_1)\,\hat{a}^{(0)}(\xi_2)\,\Gamma S^*(\xi_3)\sum_{\{\xi_3\}}\hat{a}^{(0)*}(\xi_4)\,\hat{a}^{(0)*}(\xi_5)\,\hat{a}^{(0)}(\xi_6)\\
&\quad + \dots\,.
\end{aligned} \tag{4.15}$$

West *et al.* (1978) have resumed the indicated perturbation expression via the usual lengthy and tedious diagrammatic analysis. It was found that when one retains only those diagrams that correspond to the so-called 'first Kraichman–Wyld approximation' (see e.g. Morton & Corrsin (1960), who studied the two-degrees-of-freedom Duffing oscillator diagrammatically) one obtains

$$\hat{a}^{r}(\xi) = \frac{\hat{g}_{\mathbf{k}\omega}}{\gamma_k - 2M_{\mathbf{k}}\mathscr{R}\Gamma_{\mathbf{k}} - i(\omega - \omega_k + 2M_{\mathbf{k}}\mathscr{I}\Gamma_{\mathbf{k}})}, \tag{4.16}$$

for the *renormalized* mode amplitudes as denoted by the r superscript. The function $M_{\mathbf{k}}$ appearing in the denominator of (4.16) is closely related to the normalized mean-square mode amplitude, and is given by

$$M_{\mathbf{k}} = \tfrac{1}{2}\sum_{\mathbf{l}}\langle|\hat{a}^{r}_{\mathbf{l}}(t)|^2\rangle_{r}\,(2-\delta_{\mathbf{k}-\mathbf{l}}); \tag{4.17}$$

it must be determined self-consistently using (4.16). The subscript r on the brackets in (4.17) indicates a steady-state ensemble average with respect to the renormalized steady-state distribution. The expression (4.16) for $\hat{a}^{r}(\xi)$ can be Fourier-inverted to yield

$$\frac{d}{dt}a_{\mathbf{k}}(t) + (i\omega^{r}_{k} + \gamma^{r}_{k})\,a_{\mathbf{k}}(t) = g_{\mathbf{k}}(t), \tag{4.18}$$

with

$$\omega^{r}_{k} = \omega_k - 2M_{\mathbf{k}}\mathscr{I}\Gamma_{\mathbf{k}}, \quad \gamma^{r}_{k} = \gamma_k - 2M_{\mathbf{k}}\mathscr{R}\Gamma_{\mathbf{k}}. \tag{4.19}$$

Now, in (4.3) we have assumed the fluctuations $g_{\mathbf{k}}(t)$ to be generated by a zero-centred, delta-correlated process, so that from § 3 we obtain the steady-state probability density

$$P_{ss}(\mathbf{a}) = \frac{2\gamma_{\mathbf{k}}^{l}}{\pi \mathscr{D}_{\mathbf{k}}} \exp\left\{-\frac{2\gamma_{\mathbf{k}}^{l}}{\mathscr{D}_{\mathbf{k}}}|a_{\mathbf{k}}|^2\right\}, \tag{4.20}$$

where

$$\langle g_{\mathbf{k}}(t)\, g_{\mathbf{k}'}^{*}(t-\tau)\rangle = 2\mathscr{D}_{\mathbf{k}}\,\delta_{\mathbf{k}-\mathbf{k}'}\,\delta(\tau), \quad \langle g_{\mathbf{k}}(t)\, g_{\mathbf{k}'}(t-\tau)\rangle = 0. \tag{4.21}$$

From (4.2) the linearized frequency and dissipation rate are

$$\omega_k^l = \omega_k - \mathscr{I} h_{\mathbf{k}}, \quad \gamma_k^l = \gamma_k - \mathscr{R} h_{\mathbf{k}}. \tag{4.22}$$

Thus the results (4.19) *and* (4.20) *are identical to those of* (4.2) *and* (4.3), and statistical linearization is demonstrated to be equivalent to the partial summation of an infinite sequence of perturbation diagrams. Budgor & West (1978) show that the resummation is in terms of a 'linked-cluster expansion', i.e. from all diagrams that are completely linked or connected and are therefore irreducible. The contributors to $h_{\mathbf{k}}$ are therefore the 'connected cummulants' of the expansion.

Thus the usual quasi-Gaussian approximation that is used in calculations of the moments of the gravity–capillary wave field is replaced by an approximation in which the connected cummulants at all orders enter into the calculation of the moments. In the absence of the nonlinear interactions these higher-order cummulants vanish identically in the Gaussian approximation. Therefore statistical linearization is a systematic scheme for including the effects of these non-vanishing high-order cummulants in the linearized dynamic equations.

5. The transport equation

The phase-space equation of evolution for the probability density contains the same information as the dynamic equations. In particular, the time evolution of the energy-spectral density $F(\mathbf{k},t)$ can be determined either from the linearized dynamic equations (3.13), or the Fokker–Planck equation (3.22). To see this, we multiply (3.22) on the left by $|b_{\mathbf{k}}|^2$ and integrate over all of phase space, recalling that

$$\langle|\hat{B}_{\mathbf{k}}(t)|^2\rangle = \int d\Gamma(\mathbf{b})\, |b_{\mathbf{k}}|^2 P(\mathbf{b},t\,|\,\mathbf{b}_0), \tag{5.1}$$

we obtain

$$\frac{\partial}{\partial t}\langle|\hat{B}_{\mathbf{k}}(t)|^2\rangle = \int d\Gamma(\mathbf{b})\, |b_{\mathbf{k}}|^2 \frac{\partial}{\partial t} P(\mathbf{b},t\,|\,\mathbf{b}_0), \tag{5.2}$$

since only the probability density depends explicitly on time in phase space. Therefore, after an integration of (5.2) by parts, we obtain the transport equation

$$\frac{\partial}{\partial t}F(\mathbf{k},t) + 2\alpha_{\mathrm{R}}(\mathbf{k})\,F(\mathbf{k},t) = 2D_{\mathbf{k}}. \tag{5.3}$$

A less direct, but somewhat more physical derivation of (5.3) is obtained by considering the solution to the linear dynamic equations.

The solution to (3.13) for the initial condition $\hat{B}_{\mathbf{k}}(t=0) = 0$ is

$$\hat{B}_{\mathbf{k}}(t) = \int_0^t e^{-\alpha(\mathbf{k})(t-t')} f_{\mathbf{k}}(t')\, dt', \tag{5.4}$$

and since integration is a linear process, $\hat{B}_{\mathbf{k}}(t)$ has the same statistical properties as $f_{\mathbf{k}}(t)$. Using the statistical properties of the pressure fluctuations (3.20), the solution (5.4) has a zero mean value, i.e.

$$\langle \hat{B}_{\mathbf{k}}(t) \rangle_f = 0. \tag{5.5}$$

The second moment of $\hat{B}_{\mathbf{k}}(t)$, using (2.20) and the solution (5.4), is given by

$$\langle |\hat{B}_{\mathbf{k}}(t)|^2 \rangle_f = \int_0^t dt_1 \int_0^t dt_2 \frac{\Phi(\mathbf{k}, t_1 - t_2)}{\rho_{\mathrm{W}}^2 |\mathscr{V}_k|^2} \exp\left[-\alpha(\mathbf{k})(t-t_1) - \alpha^*(\mathbf{k})(t-t_2)\right]. \tag{5.6}$$

The integration in (5.6) is handled most expeditiously by considering the time derivative of the mean-square value of $\hat{B}_{\mathbf{k}}(t)$. Using the discrete power-spectral density $F(\mathbf{k}, t)$, and taking the time derivative of (5.6), we obtain the inhomogeneous linear transport equation

$$\frac{\partial}{\partial t} F(\mathbf{k}, t) + 2\alpha_{\mathrm{R}}(\mathbf{k}) F(\mathbf{k}, t) = \frac{1}{\rho_{\mathrm{W}}^2 |\mathscr{V}_k|^2} \int_0^t d\tau\, \Phi(\mathbf{k}, \tau) \cos\left[\alpha_{\mathrm{I}}(\mathbf{k})\tau\right] \exp\left[-\alpha_{\mathrm{I}}(\mathbf{k})\tau\right]. \tag{5.7}$$

We are here interested primarily in the growth and equilibration of the gravity–capillary waves. One might guess, therefore, that the correlation time in the fluctuations of the air flow at the water surface, i.e. τ_{C}, is very much longer than the characteristic times of these waves. As argued by Phillips (1957) this might be true at very early times over time intervals in which $\Phi(\mathbf{k}, \tau)$ is essentially constant. During these intervals $\Phi(\mathbf{k}, \tau)$ may be removed from the integral (5.7). Using (5.6), the initial response of the water surface is therefore given by

$$F(\mathbf{k}, t) = \frac{\Phi(\mathbf{k}, t)}{\rho_{\mathrm{W}}^2 |\mathscr{V}_{\mathbf{k}}|^2} \frac{1}{|\alpha(\mathbf{k})|^2} \left\{\tfrac{1}{2}(1 + e^{-2\alpha_{\mathrm{R}} t}) - e^{-\alpha_{\mathrm{R}} t} \cos \alpha_{\mathrm{I}} t\right\}. \tag{5.8}$$

In the frame of reference moving with the mean wind velocity, i.e. $\alpha_{\mathrm{I}}(\mathbf{k}) \rightarrow \alpha_{\mathrm{I}}(\mathbf{k}) - \mathbf{k}.\mathbf{W}$, the surface waves in resonance with the wind are those with wave vectors such that

$$\omega_1(k) - \mathscr{I}(h_{\mathbf{k}}) = \mathbf{k}.\mathbf{W}, \tag{5.9}$$

which is the condition discussed by Phillips when $\omega_1(k)$ is replaced by ω_k and the nonlinear terms are neglected, i.e. $h_{\mathbf{k}} = 0$. Since (5.8) is only true initially, i.e. over the time interval in which $\Phi(\mathbf{k}, t)$ is constant, we assume $\alpha_{\mathrm{R}} t \ll 1$ and expand the exponential functions using (5.9) to obtain

$$F(\mathbf{k}, t) \simeq \frac{\Phi(\mathbf{k}, t)\, t^2}{\rho_{\mathrm{W}}^2 |\mathscr{V}_k|^2}, \tag{5.10}$$

which has the quadratic time dependence observed by Phillips (1957).

If we do not assume that $\Phi(\mathbf{k}, \tau)$ is frozen in the integral (5.7), but that the correlation time between fluctuations is substantially shorter than the integration time, i.e. $t \gg \tau_{\mathrm{C}}$, then, introducing the damped cosine transform of the pressure fluctuations (3.21) into (5.7), we obtain under this short-fetch condition the asymptotic inhomogeneous transport equation

$$\frac{\partial}{\partial t} F(\mathbf{k}, t) + 2\alpha_{\mathrm{R}}(\mathbf{k}) F(\mathbf{k}, t) \simeq \frac{\tilde{\Phi}_{\mathrm{C}}[\mathbf{k}, \alpha(\mathbf{k})]}{\rho_{\mathrm{W}}^2 V_k^2}, \tag{5.11}$$

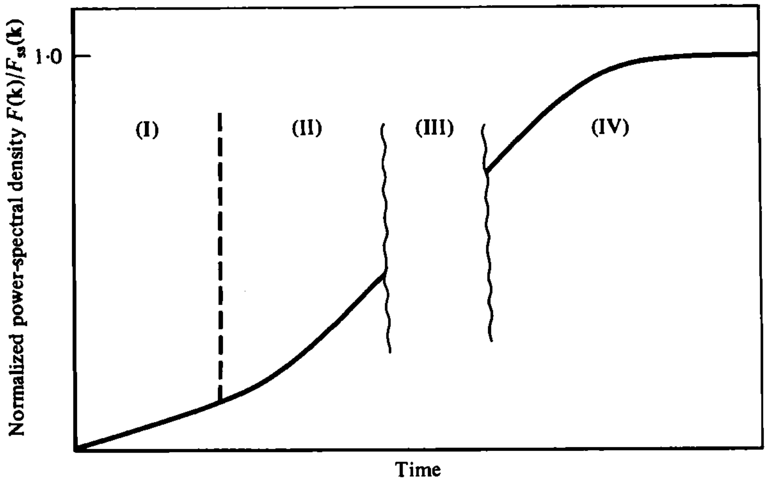

FIGURE 1. The growth as a function of time of the energy-spectral density for the gravity–capillary wave normalized to its steady-state level is indicated schematically. (I), Linear growth; (II), exponential growth; (III), transient nonlinear development; (IV), asymptotic relaxation to steady-state $F_{ss}(\mathbf{k})$.

which is precisely the expression provided by the Fokker–Planck equation, (cf. (5.3)). To solve (5.11) we must assume a spectral-form value $F(\mathbf{k}, t_a)$ at a beginning of the asymptotic time regime $t = t_a$; then by direct integration of (5.11) we obtain

$$F(\mathbf{k},t) = F(\mathbf{k},t_a)\, e^{-2\alpha_R(\mathbf{k})(t-t_a)} + \frac{\tilde{\Phi}_c[\mathbf{k},\alpha(\mathbf{k})]}{\rho_w^2 V_k^2}\, \frac{1-e^{-2\alpha_R(\mathbf{k})(t-t_a)}}{2\alpha_R(\mathbf{k})}. \tag{5.12}$$

Note that the spectral density at the beginning of this time interval decays exponentially in time. Early in this interval, i.e. for $\alpha_R(\mathbf{k})\,|t-t_a| \ll 1$, the exponential in (5.12) can be expanded as was (5.8) to obtain the linear growth in time of the surface wave spectrum beyond $F(\mathbf{k}, t_a)$, i.e.

$$F(\mathbf{k},t) \simeq F(\mathbf{k},t_a) + \frac{\tilde{\Phi}_c[\mathbf{k},\alpha(\mathbf{k})]}{\rho_w^2 V_k^2}\,(t-t_a). \tag{5.13}$$

In figure 1 we show schematically the matching of the solution (5.13) at $t = t_a$ to the presumed exponential growth of the spectral density $F(\mathbf{k}, t_a)$. The exponential growth of $F(\mathbf{k}, t_a)$ is compensated by the exponential decay $e^{-2\alpha_R(\mathbf{k})(t-t_a)}$ resulting in a *net* relaxation of this 'initial condition' as determined by the real part of $h_\mathbf{k}$. This is the quenching of the instability in the Miles–Phillips model owing to the average nonlinear interactions.

The steady-state power-spectral density predicted by the solution (5.13) is given by the $t \to \infty$ value, i.e.

$$F_{ss}(\mathbf{k}) \equiv \lim_{t\to\infty} F(\mathbf{k},t) = \frac{\tilde{\Phi}_c[\mathbf{k},\alpha(\mathbf{k})]}{2\rho_w^2 V_k^2 \alpha_R(\mathbf{k})} \tag{5.14}$$

as given by (3.27), and depends on the average nonlinear interactions through the variational parameter $h_\mathbf{k}$ contained in $\alpha_R(\mathbf{k})$. Therefore, in determining the relaxation rate $\alpha_R(\mathbf{k})$ we also determine the steady-state level of the gravity–capillary wave spectrum. In § 3 we found that the parameter $h_\mathbf{k}$ is in turn evaluated using the steady-state spectral density $F_{ss}(\mathbf{k})$. A self-consistent prescription for the steady-state power-spectral density $F_{ss}(\mathbf{k})$ and the average nonlinear interactions in the gravity–capillary spectrum $h_{\mathbf{k}}$ is given by (3.27) and (3.29).

Our model includes three regions of growth (depicted in figure 1) for the power-spectral density of the surface wave field. For early times, region (I) in figure 1, we have the linear growth regime of Phillips, with the growth rate proportional to the power spectral density of the turbulent fluctuations in the wind field $\Phi[\mathbf{k}, \alpha'(\mathbf{k})]$. The prime on $\alpha'(\mathbf{k})$ is used here as a reminder that perturbation theory must be used to determine the early-time behaviour of the mode amplitudes. For intermediate times, there is some region of exponential growth as prescribed by the Miles instability mechanism. This growth is indicated in region (II) of figure 1. The asymptotic, or long-time region, gives the steady-state or saturation spectrum $F_{ss}(\mathbf{k})$ in region (IV) of figure 1. The steady-state spectral level is determined by $\tilde{\Phi}_c[\mathbf{k}, \alpha(\mathbf{k})]$, and $\alpha(\mathbf{k})$ which depend implicitly on the average nonlinear hydrodynamic interactions through the parameter $h_{\mathbf{k}}$ which we calculate in § 6. The connection between regions (II) and (IV) in figure 1 is left tenuous because in region (III) the nonlinear interactions are developing and their transient behaviour remains undescribed.

6. Calculation of $F_{ss}(\mathbf{k})$ and $h_{\mathbf{k}}$

In this section we calculate the steady-state energy-spectral density of the gravity–capillary waves $F_{ss}(\mathbf{k})$ and the asymptotic relaxation rate of these waves from $h_{\mathbf{k}}$. The expressions for these two quantities are

$$F_{ss}(\mathbf{k}) = \frac{\tilde{\Phi}_c[\mathbf{k}, \alpha(\mathbf{k})]}{2\rho_w^2 V_k^2 \mathscr{R}[\lambda_k - h_{\mathbf{k}}]} \tag{6.1a}$$

$$h_{\mathbf{k}} = \frac{2}{\rho_w^2} \sum_{\mathbf{l}} \frac{C_{\mathbf{kl}}^{\mathbf{kl}}}{V_l^2} \frac{\tilde{\Phi}_c[\mathbf{l}, \alpha(\mathbf{l})]}{\mathscr{R}[\lambda_k - h_{\mathbf{k}}]}. \tag{6.1b}$$

In the continuum limit the discrete steady-state spectrum $F_{ss}(\mathbf{k})$ is replaced by the continuum spectrum $\Psi_{ss}(\mathbf{k})$ using

$$F_{ss}(\mathbf{k}) = \frac{\Sigma_0}{(2\pi)^2} \Psi_{ss}(\mathbf{k}), \tag{6.2}$$

and the discrete sum is replaced by an integral over wave vector,

$$\sum_{\mathbf{k}} \rightarrow \frac{\Sigma_0}{(2\pi)^2} \int d^2k, \tag{6.3}$$

so that (6.1 *b*) becomes

$$h_{\mathbf{k}} = 4 \int d^2 l\, \bar{C}_{\mathbf{kl}}^{\mathbf{kl}} \Psi_{ss}(\mathbf{l}). \tag{6.4}$$

To evaluate the integral (6.4) we must have a steady-state spectrum for the gravity–capillary waves $\Psi(\mathbf{l})$. The steady-state solution to the transport equation (5.11) is given by (6.1 *a*), so we identify the inhomogeneous term in (5.11) with the functional form for the observed pressure-fluctuation power-spectral density proposed by Phillips (1977, § 4.2). Although the approximation arguments presented by Phillips applied to longer-wavelength waves we use $\tilde{\Phi}_c[\mathbf{k}, \alpha(\mathbf{k})] = \pi\Pi(\mathbf{k}, n)\Sigma_0/(2\pi)^2$ so that

$$(2\pi)^2 \tilde{\Phi}_c(\mathbf{k}, \alpha)/\Sigma_0 = 20\Pi(\omega_k)/2\pi k^2, \tag{6.5}$$

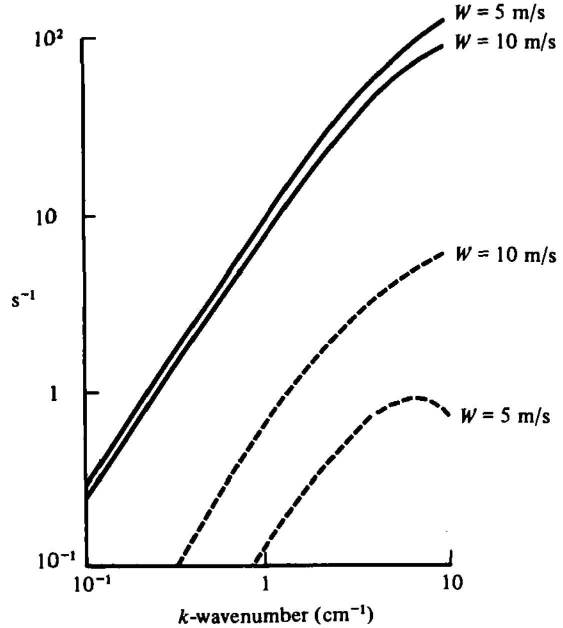

FIGURE 2. The magnitude of the initial growth rates of the gravity–capillary waves as a function of frequency (– – – –) for wind speeds of 5 m/s and 10 m/s are contrasted with the relaxation rates in the asymptotic-time domain (——).

where $\Pi(\omega_k)$ is an experimental spectral density for the pressure fluctuations as a function of frequency. We use the measurements of Elliott (1972) in the form presented in figure 4.2 of Phillips (1977), to write

$$\Pi(\omega_k) \simeq \frac{1{\cdot}5\times 10^{-3}\, 2\pi\rho_a^2}{(22{\cdot}5)^4}\frac{W^5}{\omega_k^2}, \tag{6.6}$$

where we have used $W = 22{\cdot}5U_*$, and U_* is the wind friction velocity. Using (6.5) and (6.6) in (6.1 a) we write the steady-state power-spectral density of the water wave field (6.2) as

$$\begin{aligned}\Psi_{ss}(\mathbf{k}) &= 3{\cdot}68\times 10^{-6}\left(\frac{\rho_a}{\rho_w}\right)^2 \frac{W^5}{k^2\omega_k^2|\mathscr{V}_k|^2}\frac{1}{2\alpha_R(\mathbf{k})}\\ &= 3{\cdot}68\times 10^{-12}\frac{W^5}{\omega_k^4}\frac{1}{2\alpha_R(\mathbf{k})}.\end{aligned} \tag{6.7}$$

Note that (6.4), with (6.7) as the energy-spectral density for the surface-wave field, has the functional form

$$h_{\mathbf{k}} = \int d^2l \frac{G(\mathbf{k},\mathbf{l})}{\mathscr{R}[\lambda_l - h_l]}, \tag{6.8}$$

which is an integral equation for $h_{\mathbf{k}}$. We solve this equation by first making a guess at the functional form of $h_{\mathbf{k}}$ and then iterating until the value of $h_{\mathbf{k}}$ converges. The results of such calculations are shown in figure 2.

In these calculations we use the value of the air–sea coupling parameter that was found to agree quite well with the laboratory experiments of Plant & Wright (1977), i.e.

$$\mu_k \simeq 2\times 10^{-5}\left(\frac{W}{V_k}\right)^2, \tag{6.9}$$

Wavenumber (cm^{-1})	Frequency (Hz)	Normalized spectrum $f^4\Phi(f)$ W = 500 cm/s	W = 10^3 cm/s
10	46·4	33	316
7·57	31·8	28	275
5·72	22·3	26	254
4·33	16·2	28	265
3·28	12·18	31	299
2·48	9·5	39	369

TABLE 1.

where W is the wind speed determined from the wind friction velocity as in (6.6), and V_k is the phase speed of the linear surface wave. In figure 2 we compare the growth rate predicted by

$$|\beta_k| = 2|\mu_k \omega_k - \nu k^2|, \tag{6.10}$$

for wind speeds of 10 m/s and 5 m/s with the calculated asymptotic relaxation rates for the same wind speeds.

It is apparent from figure 2 that perturbations of the high-frequency wave spectrum relax to the steady state, i.e. are damped out, at a rate which is one to two orders of magnitude faster than the initial exponential rate of growth of that portion of the spectrum. The nonlinear interactions are therefore very efficient in transferring energy out of the perturbed region once the waves have reached their steady-state levels, i.e., efficient in establishing an energy cascade in the steady state. This energy is cascaded from longer to shorter waves, where it is viscously damped. As an example of the rate of the relaxation consider a 6 mm wavelength wave that has an initial growth rate of 5·6 s^{-1} as given by (6.10) for a 10 m/s wind speed. A perturbation of this wave near the steady state, however, relaxes to the steady-state level at a calculated rate of 84 s^{-1}, i.e. the perturbation vanishes in approximately one half of a cycle of the wave.

In table 1 the calculated spectrum for the gravity–capillary waves is given for six frequencies in the interval 10–50 Hz. In this range of frequencies the spectrum normalized by f^4 seems to be fairly constant for both wind speeds of 5 m/s and 10 m/s. However, the level of the spectrum is strongly wind-dependent, as is found in the experiments of Lleonart & Blackman (1980). The dependence calculated here is approximately a factor of two stronger than they observed experimentally. The experimental slope of the spectrum was determined to be f^{-3} in this frequency range, whereas we calculate it to be f^{-4}, a slope which is observed experimentally at somewhat higher frequencies. The frequency at which the spectrum changes from a f^{-3} to a f^{-4} frequency dependence is wind-speed dependent, moving to higher frequencies for higher wind speeds.

The major discrepancy is not completely unexpected, since the coupling to the longer-wavelength gravity waves becomes an important mechanism for these lower-frequency waves. This nonlinear coupling, which has *not* been included in the present model, transfers energy out of this spectral interval as found experimentally by Plant & Wright (1977), thereby depleting the energy in the 10 Hz and lower frequency ranges. This mechanism thus decreases the steepness of the measured slopes of the frequency spectrum. A model including this mechanism is presently being studied.

7. Discussion and conclusion

The linear-coupling model of the air–sea interaction developed by Miles (1957, 1960) and Phillips (1960) has been generalized to include the average nonlinear interaction in the dynamic equations. We have demonstrated by direct integration of these dynamic equations that the average nonlinear interactions for gravity–capillary waves do indeed quench the instability in the growth of water waves predicted by the Miles–Phillips model in this region of the spectrum. The central assumption in the present model is the applicability of the statistical-linearization method in the treatment of the nonlinear interactions in the asymptotic regime. The value of this technique in studying nonlinear systems with many degrees of freedom has been discussed elsewhere, see e.g. Budgor & West (1978), West *et al.* (1978) and West (1980).

The linearized equations provide a simple Langevin model to describe the air–sea interaction in which the nonlinear interactions among the water waves provide the 'dissipation' required to establish a steady state in the energy-spectral density. In statistical mechanics the Langevin equation is often used to describe systems near thermal equilibrium (see e.g. de Groot & Mazur 1969) in which the fluctuations are spatially correlated with their relaxation, resulting in a simple fluctuation–dissipation relation. In the water-wave field the energy is transported from large (approximately 10 cm) to small scales by the nonlinear interactions among the fluctuating water waves, where it is viscously damped. This energy cascade results in a lack of correlation between fluctuations and their relaxation and the consequent lack of a simple fluctuation–dissipation relation in this case. There is, however, a nonlinear integral relation, given by (6.1), relating the energy supplied to the water-wave field at all scales by the fluctuating wind field and its subsequent dissipation at the short scales. This relation determines the existence of an asymptotic steady state.

The calculations indicate that perturbations of the short waves relax to their steady-state levels one to two orders of magnitude faster than the rate at which they initially grow, and unlike their initial growth rates their relaxation rates are fairly insensitive to wind speed. The calculated energy-spectral density decreases more rapidly with frequency at longer wavelengths than observed experimentally by Lleonard & Blackman (1980). This is probably a result of not including the coupling of the gravity–capillary waves to the gravity waves; a mechanism that Plant & Wright (1977) find very important in their experiments.

The second key assumption made in the analysis is the statistical properties of the pressure-field fluctuations and the form of the power-spectral density of these fluctuations. These assumptions, although not crucial, were necessary for obtaining a closed-form analytic expression for the power-spectral density of the surface waves. Approximate treatments for non-Gaussian statistics or long correlation times can be developed as generalizations of the present model, but more detailed assumptions would require a more extensive data base in the pressure-field fluctuations than presently exists. The calculated values of the relaxation rates and the steady-state spectral densities are related directly to the experimental information available on the pressure-field fluctuations.

I would like to acknowledge the many fruitful discussions with V. Seshadri and K. Lindenberg on many of the techniques applied in this work.

Appendix A. Equations of motion

The equations of motion (2.1) expressed in terms of quantities defined on the fluid surface are

$$\left.\begin{aligned}\frac{\partial\phi_s}{\partial t}+\tfrac{1}{2}(\nabla_x\phi_s)^2+g\zeta-[\gamma-\tfrac{3}{2}\gamma(\nabla_x\zeta)^2-2\nu W]\nabla_x^2\zeta-2\nu\nabla_x^2\phi_s=\tfrac{1}{2}W^2[1+(\nabla\zeta)^2]-p/p_w,\\ \frac{\partial\zeta}{\partial t}+\nabla_x\phi_s.\nabla_x\zeta=[1+(\nabla_x\zeta)^2]\,W,\end{aligned}\right\} \tag{A 1}$$

where the vertical velocity at the surface is

$$W\equiv\left.\frac{\partial\phi}{\partial z}\right|_{z=\zeta}. \tag{A 2}$$

We express the equations of motion (2.1) in terms of quantities defined on the free surface $z=\zeta(\mathbf{x},t)$ to avoid the limitations of performing an expansion about $z=0$ for high-frequency waves. The restrictions on the $z=0$ expansion were pointed out by a number of investigators, including Watson & West (1975) and Holliday (1977). We therefore present in this appendix the technique developed by Watson & West for expanding the vertical velocity (A 2) about the $z=\zeta(\mathbf{x},t)$ surface, but now applied to gravity–capillary waves.

The method we employ in the analysis of the vertical velocity W in (A 1) is a special application of potential theory with Dirichlet boundary conditions used by Watson & West (1975). To apply the method we introduce the velocity potential on the reference plane $z=0$ and define

$$\phi_0(\mathbf{x},t)\equiv\phi(\mathbf{x},z,t)|_{z=0}. \tag{A 3}$$

Since specification of the potential on a closed surface defines a unique potential problem we assert that the velocity potential at the free surface can be written as a Taylor-series expansion about the $z=0$ plane:

$$\phi_s(\mathbf{x},t)=\sum_{n=0}^{\infty}\frac{\zeta^n}{n!}\left.\frac{\partial^n\phi}{\partial z^n}\right|_{z=0}. \tag{A 4}$$

The vertical derivatives in (A 4) can be replaced by the operator κ using the fact that $\phi(\mathbf{x},z,t)$ satisfies Laplace's equation in the fluid interior, i.e.

$$\nabla^2\phi=0\Rightarrow\kappa^2\phi=-\nabla_x^2\phi. \tag{A 5}$$

Formally, the operator κ replaces $\partial/\partial z$ in Laplace's equation and has the form given in (2.4), $\kappa\equiv(-\nabla_x^2)^{\frac{1}{2}}$, enabling us to replace a vertical-derivative operation with a horizontal-derivative operation and to delete reference to the z-co-ordinate altogether. The quantity κ is defined to operate only on Fourier series such that if an arbitrary function $f(\mathbf{x})$ has a Fourier-series expansion

$$f(\mathbf{x})=\sum_{\mathbf{k}}f_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}},$$

then operating with κ yields

$$\kappa f(\mathbf{x})=\sum_{\mathbf{k}}kf_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}.$$

The velocity potential at the free surface (A 4) can be expressed in terms of the operator $\mathcal{O}(\mathbf{x},t)$ defined by the series expansion

$$\phi_s(\mathbf{x},t) = \sum_{n=0}^{\infty} \mathcal{O}_{\mathbf{k}}\phi_0(\mathbf{x},t) = \mathcal{O}(\mathbf{x},t)\,\phi_0(\mathbf{x},t), \tag{A 6}$$

where

$$\mathcal{O}_n \equiv \frac{1}{n!}\zeta^n(\mathbf{x},t)\,\kappa^n, \tag{A 7}$$

and is obtained by replacing $\partial/\partial z$ by κ in (A 4). In a similar manner we can write the vertical fluid velocity as a Taylor series with an additional vertical derivative:

$$W(\mathbf{x},t) = \sum_{n=0}^{\infty} Q_n\phi_0(\mathbf{x},t) = Q(\mathbf{x},t)\,\phi_0(\mathbf{x},t), \tag{A 8}$$

where

$$Q_n \equiv \frac{1}{n!}\zeta^n(\mathbf{x},t)\,\kappa^{n+1}. \tag{A 9}$$

Now to express W in terms of ϕ_s we need only express the intermediary velocity potential ϕ_0 in terms of ϕ_s by inverting the operator $\mathcal{O}(\mathbf{x},t)$ in (A 6), i.e.

$$\phi_0(\mathbf{x},t) = \mathcal{O}^{-1}(\mathbf{x},t)\,\phi_s(\mathbf{x},t), \tag{A 10}$$

and write (A 8) as

$$W(\mathbf{x},t) = Q(\mathbf{x},t)\,\mathcal{O}^{-1}(\mathbf{x},t)\,\phi_s(\mathbf{x},t). \tag{A 11}$$

The operator $\mathcal{O}(\mathbf{x},t)$ represents the projection of the velocity potential defined on the $z = 0$ reference plane onto the free surface. The inverse operation $\mathcal{O}^{-1}(\mathbf{x},t)$ projects the velocity potential from the free surface back into the reference plane. For moderate surface deviations the inverse operator $\mathcal{O}^{-1}$ can be expressed as a perturbation series in the surface displacement, i.e.

$$\mathcal{O}^{-1}(\mathbf{x},t) = 1-\mathcal{O}_1(\mathbf{x},t)-\mathcal{O}_2(\mathbf{x},t)+\mathcal{O}_1^2(\mathbf{x},t)+\mathcal{O}_2^2(\mathbf{x},t) + \mathcal{O}_1(\mathbf{x},t)\,\mathcal{O}_2(\mathbf{x},t)+\mathcal{O}_2(\mathbf{x},t)\,\mathcal{O}_1(\mathbf{x},t)+\dots . \tag{A 12}$$

The vertical velocity (A 11) can then be written

$$W = Q_0\phi_s + (Q_1 - Q_0\mathcal{O}_1)\,\phi_s + (Q_2 - Q_1\mathcal{O}_1 - Q_0\mathcal{O}_1^2)\,\phi_s - (Q_2\mathcal{O}_1 + Q_1\mathcal{O}_2 - Q_1\mathcal{O}_1^2 - Q_0\mathcal{O}_1\mathcal{O}_2 - Q_1\mathcal{O}_2\mathcal{O}_1)\,\phi_s + \dots, \tag{A 13}$$

to third order in the vertical surface displacement. A general series expression of terms of $\mathcal{O}_n$ and Q_n can be written for the vertical velocity, but this is not of interest to us here. Direct substitution of the defining equations (A 7) and (A 9) into (A 13) yields the vertical velocity

$$W = \kappa\phi_s - [\kappa(\zeta\kappa\phi_s) - \zeta\kappa^2\phi_s] + \kappa[\zeta\kappa(\zeta\kappa\phi_s)] - \zeta[\kappa^2(\zeta\kappa\phi_s)] - \tfrac{1}{2}[\kappa(\zeta^2\kappa^2\phi_s) - \zeta^2\kappa^3\phi_s] + \dots . \tag{A 14}$$

The perturbative equations of motion at the free surface are, to third order,

$$\frac{\partial\phi_s}{\partial t} + g\zeta \simeq \tfrac{1}{2}[(\kappa\phi_s)^2 - (\nabla_{\mathbf{x}}\phi_s)^2] + \kappa\phi_s[\zeta\kappa^2\phi_s - \kappa(\zeta\kappa\phi_s)] - p/\rho_w + \gamma[1-\tfrac{3}{2}(\nabla_{\mathbf{x}}\zeta)^2]\,\nabla_{\mathbf{x}}^2\zeta + 2\nu[\nabla_{\mathbf{x}}^2\phi_s - \nabla_{\mathbf{x}}^2\zeta(\kappa\phi_s - \kappa(\zeta\kappa\phi_s) + \zeta\kappa^2\phi_s)] + \dots, \tag{A 15a}$$

$$\frac{\partial\zeta}{\partial t} - \kappa\phi_s \simeq -\nabla_{\mathbf{x}}\phi_s.\nabla_{\mathbf{x}}\zeta - \kappa(\zeta\kappa\phi_s) + \zeta\kappa^2\phi_s + \kappa\phi_s(\nabla_{\mathbf{x}}\zeta)^2 + \kappa[\zeta\kappa(\zeta\kappa\phi_s)] - \tfrac{1}{2}\zeta^2\kappa^2\phi_s - \zeta\kappa^2(\zeta\kappa\phi_s) + \tfrac{1}{2}\zeta^2\kappa^3\phi_s + \dots . \tag{A 15b}$$

The nonlinear terms in (A 15a, b) give rise to the nonlinear functions F_ϕ and F_ζ, respectively.

Appendix B. Interaction coefficients

The second-order interaction coefficients in (2.16) are given by

$$C^{\mathbf{k}}_{\mathbf{lm}} = \Gamma^{\mathbf{k}}_{\mathbf{lm}} - \tfrac{1}{4}i\mu_k[(lk-\mathbf{l}.\mathbf{k})\mathscr{V}_l + (mk-\mathbf{m}.\mathbf{k})\mathscr{V}_m] + \frac{i\nu}{4\mathscr{V}_k}[l^2m\mathscr{V}_m + m^2l\mathscr{V}_l], \quad \text{(B 1}a\text{)}$$

$$C^{\mathbf{km}}_{\mathbf{l}} = \Gamma^{\mathbf{km}}_{\mathbf{l}} + \tfrac{1}{4}i\mu_k[(lk-\mathbf{l}.\mathbf{k})\mathscr{V}_l - (km+\mathbf{k}.\mathbf{m})\mathscr{V}^*_m] + \frac{i\nu}{4\mathscr{V}_k}[l^2m(\mathscr{V}^*_m - \mathscr{V}_m) + m^2l(\mathscr{V}^*_l - \mathscr{V}_l)], \quad \text{(B 1}b\text{)}$$

$$C^{\mathbf{klm}} = \Gamma^{\mathbf{klm}} + \tfrac{1}{4}i\mu_k[(kl+\mathbf{k}.\mathbf{l})\mathscr{V}^*_l + (km+\mathbf{k}.\mathbf{m})\mathscr{V}^*_m] - \frac{i\nu}{4\mathscr{V}_k}[ml^2\mathscr{V}^*_m + lm^2\mathscr{V}^*_l], \quad \text{(B 1}c\text{)}$$

where the *form* of the Γs are given in Watson & West (1975) as

$$\Gamma^{\mathbf{k}}_{\mathbf{lm}} = \frac{1}{8}\left[\frac{\mathscr{V}_l\mathscr{V}_m}{\mathscr{V}_k}(lm+\mathbf{l}.\mathbf{m}) - \mathscr{V}_l(lk-\mathbf{l}.\mathbf{k}) - \mathscr{V}_m(mk-\mathbf{m}.\mathbf{k})\right]. \quad \text{(B 2}a\text{)}$$

$$\Gamma^{\mathbf{km}}_{\mathbf{l}} = \frac{1}{4}\left[\frac{\mathscr{V}_l\mathscr{V}^*_m}{\mathscr{V}_k}(lm-\mathbf{l}.\mathbf{m}) + \mathscr{V}_l(lk-\mathbf{l}.\mathbf{k}) - \mathscr{V}^*_l(mk+\mathbf{m}.\mathbf{k})\right], \quad \text{(B 2}b\text{)}$$

$$\Gamma^{\mathbf{klm}} = \frac{1}{8}\left[\frac{\mathscr{V}^*_l\mathscr{V}^*_m}{\mathscr{V}_k}(lm+\mathbf{l}.\mathbf{m}) + \mathscr{V}^*_l(lk+\mathbf{l}.\mathbf{k}) + \mathscr{V}^*_m(mk+\mathbf{m}.\mathbf{k})\right]. \quad \text{(B 2}c\text{)}$$

We note that $\mathscr{V}_k$ is the 'complex velocity' obtained from (2.13) to be

$$\mathscr{V}_k = [V^2_k - (\mu_k V_k - \nu k)^2]^{\frac{1}{2}} + i(\mu_k V_k - \nu k), \quad \text{(B 3)}$$

where $V_k\,(= (g/k+\gamma k)^{\frac{1}{2}})$ is the phase velocity of the linear wave.

The direct contribution of the quadratic interactions to the steady-state value of the wave energy-spectral density through $h_{\mathbf{k}}$ vanishes owing to the resonant restrictions on wave vectors. Here we modify thę third-order interaction coefficients as was done in Watson & West (1975) to include these quadratic effects in a perturbative manner, i.e.

$$\bar{C}^{\mathbf{kn}}_{\mathbf{lm}} = C^{\mathbf{kn}}_{\mathbf{lm}} + \frac{C^{\mathbf{k}}_{\mathbf{l},\mathbf{m}-\mathbf{n}}C^{\mathbf{m}-\mathbf{n},\mathbf{n}}_{\mathbf{m}}}{\lambda_{\mathbf{m}-\mathbf{n}} - \lambda_m - \lambda^*_n} + \frac{C^{\mathbf{k}}_{\mathbf{m},\mathbf{l}-\mathbf{n}}C^{\mathbf{l}-\mathbf{n},\mathbf{n}*}_{\mathbf{l}}}{\lambda_{\mathbf{l}-\mathbf{n}} - \lambda_l - \lambda^*_n} - \frac{1}{2}\frac{C^{\mathbf{k},\mathbf{n}-\mathbf{m}}_{\mathbf{l}}C^{\mathbf{n}-\mathbf{m},\mathbf{m}}_{\mathbf{n}}}{\lambda_m + \lambda^*_n - \lambda^*_{\mathbf{k}-\mathbf{l}}} - \frac{1}{2}\frac{C^{\mathbf{k},\mathbf{n}-\mathbf{l}}_{\mathbf{m}}C^{\mathbf{n}-\mathbf{l},\mathbf{l}}_{\mathbf{n}}}{\lambda_l + \lambda^*_n - \lambda^*_{\mathbf{k}-\mathbf{m}}}$$
$$+ \frac{C^{\mathbf{k},\mathbf{n}}_{\mathbf{l}+\mathbf{m}}C^{\mathbf{l}+\mathbf{m}}_{\mathbf{l},\mathbf{m}}}{\lambda_{\mathbf{l}+\mathbf{m}} - \lambda_m - \lambda_l} + 2\frac{C^{\mathbf{k},\mathbf{n},-(\mathbf{l}+\mathbf{m})}C^{-(\mathbf{l}+\mathbf{m}),\mathbf{l},\mathbf{m}*}}{\lambda^*_{\mathbf{l}+\mathbf{m}} - \lambda_m - \lambda_l}, \quad \text{(B 4)}$$

where

$$C^{\mathbf{kn}}_{\mathbf{lm}} = \frac{i}{16}k\Big\{n\mathscr{V}_n[|\mathbf{l}-\mathbf{n}| + |\mathbf{m}-\mathbf{n}| - k - n] + l\mathscr{V}_l[k+l-|\mathbf{l}-\mathbf{n}| - |\mathbf{k}+\mathbf{n}|]$$
$$+ m\mathscr{V}_m[k+m-|\mathbf{m}-\mathbf{n}| - |\mathbf{k}+\mathbf{n}|] - \frac{lm}{k}\frac{\mathscr{V}_l\mathscr{V}_m}{\mathscr{V}_k}[l+m-|\mathbf{l}-\mathbf{n}| - |\mathbf{m}-\mathbf{n}|]$$
$$+ \frac{nm}{k}\frac{\mathscr{V}_n\mathscr{V}_m}{\mathscr{V}_k}[n+m-|\mathbf{l}-\mathbf{n}| - |\mathbf{k}+\mathbf{n}|] + \frac{nl}{k}\frac{\mathscr{V}_n\mathscr{V}_l}{\mathscr{V}_k}[l+n-|\mathbf{m}-\mathbf{n}| - |\mathbf{k}+\mathbf{n}|]\Big\}. \quad \text{(B 5)}$$

We have neglected the direct dependence of the third-order coupling coefficient on γ and ν, and an error in the third-order coefficients $\Gamma^{\mathbf{kn}}_{\mathbf{lm}}$ in Watson & West (1975) has been corrected. The second-order terms (three-wave interactions) generate four-wave interactions in (B 4) and modify the coupling strength.

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